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Diagram and superfield techniques in the classical superalgebras

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Abstract. We introduce the concept of 'graded permutation group' in the analysis of tensor operators in the classical superalgebras. For $U(m/n)$ and $SU(m/n)$, irreducible tensor representations correspond to classes of Young tableaux with definite graded symmetry type. Diagram techniques are given for Kronecker products, dimensions, and branching rules such as $U(m+\mu/n+\nu)\supset U(m/n)\times U(\mu/\nu)$ and $U(m\mu+n\nu/m\nu+n\mu)\supset U(m/n)\times U(\mu/\nu)$.

The tensor techniques are complemented by the introduction of a superfield formalism, in which $U(m/n)$ and $SU(m/n)$ act on (polynomial) functions over the appropriate superspace. Such superfields may admit constraints. A general superfield interpolates between the classes of Young tableaux which correspond to particular types of constraint. The tensor and superfield techniques are illustrated with case studies of $SU(2/1)$ and $SU(m//1)$.

1. Introduction

The motivation for an investigation of the superalgebras or graded algebras stems ultimately from the realisation of their widespread application in mathematics and physics (Corwin *et al* 1975). The present work is a contribution to the study of the representations of the classical superalgebras, in particular $U(m/n)$ and $SU(m/n)$. It is aimed at making available a range of techniques which are well established for the classical Lie algebras, and readily suited to applications, but which hitherto have been somewhat neglected in the superalgebra case. These techniques are the complementary ones of tensor and differential (or superfield) realisations.

In the subject of supersymmetry in space–time (for a review, see Fayet and Ferrara 1977), the early work (Ferrara *et al* 1974, Salam and Strathdee 1974) has been traditionally concerned with superfield formulations for the Poincaré superalgebra and its N-extended generalisations (Dondi 1975, Salam and Strathdee 1975), with some attention to the conformal (Dondi and Sohnius 1974), and de Sitter (Keck 1975) cases, and other non-superfield studies of the unitary representations in the Poincaré case (Jarvis 1976, 1977, Grisaru 1977).

With the advent of the supergravity theories (for a review, see van Nieuwenhuizen and Freedman 1979), there has been increased incentive for the study of the representations of the N-extended Poincaré superalgebra (Fayet 1976, Sohnius 1978), and construction of the corresponding superfields. The superfield representations of the de

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Sitter superalgebra and its extensions (Keck 1975, Ivanov and Sorin 1979, 1980) are important for the formulations of extended supergravity constraints (Gates *et al* 1979).

Recently, the possibility of using an internal superalgebra as a gauge symmetry has been investigated (Wouthuysen 1978, Ne'eman 1979, Fairlie 1979, Dondi and Jarvis 1979, Taylor 1979a). Candidates such as SU(5/1) have also been considered (Dondi and Jarvis 1979, Taylor 1979b, c); a knowledge of the irreducible representations of the superalgebras has naturally been required for these applications. Some preliminary results of the present work have been given elsewhere (Dondi and Jarvis 1980).

In parallel with such physical applications, the mathematical theory of Lie superalgebras has been developed considerably (Kac 1977, Scheunert 1979 and references therein; for a review, see Rittenberg 1978). The representation theory has been studied from the point of view of the general theory, and of simple examples such as sl(2/1) and osp(1/2) (Scheunert *et al* 1977, Marcu 1980a, b), or osp($1/n$) (Corwin 1976, Bednar and Sachl 1978). Issues such as the existence of the characteristic identities for gl(m/n), sl(m/n) and osp(m/n) have been followed up (Jarvis and Green 1979).

Nonetheless, a unified treatment of the tensor representations, of the sort familiar from many standard texts for the classical Lie groups (Weyl 1939, Hamermesh 1962), has not been given before. In the present paper (concentrating here on U(m/n) and SU(m/n)), we show that the concept of a tensor representation is possible in the superalgebra case, and that the usual connection between symmetrised tensors of rank r and the permutation group on r symbols continues to hold, with due allowances for modifying sign factors arising from the grading. Therefore, for a large class of representations, the usual S-function (or Young diagram) techniques for Kronecker products, branching rules, dimension formulae, plethysyms, and so on, which rely solely on the nature of the permutation group, can be transferred (with suitable modifications) to the graded case (§ 2). The work relies heavily on Jarvis and Green (1979) for the basic tensor operator formalism (for the gl(m/n), sl(m/n) and osp(m/n) cases); a similar formalism can be developed for the remaining classical superalgebras p(m) and q(m), and will be given elsewhere.

These tensor techniques are complemented by the development of a corresponding superfield formalism (§ 3). As usual, the basic ingredients are the little group (the even part of the superalgebra, for example U(m) × U(n)), and the corresponding coset space, or superspace. Superfields are functions over superspace taking values in the carrier space of a representation of the little group, and on which the whole supergroup acts (see, for example, Dondi and Sohnius 1974). This action may in general be decomposed by applying certain constraints, some of which are shown to correspond to the classes of irreducible symmetrised tensors. In general, however, the superfields interpolate between the irreducible tensor representations.

Sections 2 and 3 end with a discussion of SU(m/1) and a case study of SU(2/1), where both the irreducible tensors and constraints are easily found, and can be compared with results in the literature (Scheunert *et al* 1977, Marcu 1980a, b). Further comments and concluding remarks are made in § 4.

2. Graded Young diagrams

The concept of a tensor representation in the classical superalgebras follows naturally from the tensor operator formalism of Jarvis and Green (1979), which we summarise here for convenience.

The $(m+n)^2$ generators of $\mathrm{gl}(m/n)$ or $\mathrm{U}(m/n)$ satisfy the commutation and anticommutation relations

$$[E^A{}_B, E^C{}_D]_{-\left[\begin{smallmatrix}AC\\BD\end{smallmatrix}\right]} = \delta^C{}_B E^A{}_D - \begin{bmatrix}AC\\BD\end{bmatrix} \delta^A{}_D E^C{}_B \tag{1}$$

where $A, B, \ldots = 1, 2, \ldots, m+n$. Here indices in the range $a, b, \ldots = 1, \ldots, m$ are called 'even', and assigned a grading $(a) = (b) = \ldots = 0$, and indices in the range $\alpha, \beta = m+1, \ldots, m+n$ are called 'odd', and graded $(\alpha) = (\beta) = \ldots = 1$. A generator is 'even' or 'odd' according to its index structure: thus $E^A{}_B$ is graded $(A)+(B) \equiv (A+B)$, while the bracket of $E^A{}_B$ and $E^C{}_D$ becomes an anticommutator whenever $(A+B)(C+D) \equiv 1 (\mathrm{mod}\ 2)$, as expressed by the sign factor $\left[\begin{smallmatrix}AC\\BD\end{smallmatrix}\right] = (-1)^{(A+B)(C+D)}$. General sign functions of several indices are simiarly interpreted as a product of column sums in the exponent.

If $m \neq n$, the $(m+n)^2 - 1$ generators defined by

$$A^A{}_B = E^A{}_B - \frac{1}{m-n} \delta^A{}_B [B](E^X{}_X), \tag{2}$$

with summation on repeated indices, satisfy the same superalgebra as $\mathrm{gl}(m/n)$, and generate the simple subalgebra $\mathrm{sl}(m/n)$ or $\mathrm{SU}(m/n)$. If $m = n$, the $\mathrm{gl}(m/n)$ formalism is unchanged, but the above definition of the $\mathrm{sl}(m/n)$ generators is inapplicable.

The commutation and anticommutation rules (1) and (2) suggest that a vector operator, say X_C or X^C, can be defined by the following transformation laws:

$$[E^A{}_B, X_C]_{-\left[\begin{smallmatrix}A\\B\end{smallmatrix}C\right]} = -\begin{bmatrix}A\\B\end{bmatrix}C \delta^A{}_C X_B$$

and

$$[E^A{}_B, X^C]_{-\left[\begin{smallmatrix}A\\B\end{smallmatrix}C\right]} = +\delta^C{}_B X^A \tag{3}$$

and similarly by considering $X^{A_1}X^{A_2}, X^{A_1}X^{A_2}X^{A_3}, \ldots$, a (contravariant) tensor operator of arbitrary rank will transform as

$$[E^A{}_B, X^{A_1A_2A_3\cdots}]_{-\left[\begin{smallmatrix}AA_1\\BA_2\\A_3\end{smallmatrix}\right]} = \delta^{A_1}{}_B X^{AA_2A_3\cdots} + \delta^{A_2}{}_B \begin{bmatrix}AA_1\\B\end{bmatrix} X^{A_1AA_3\cdots}$$
$$+ \delta^{A_3}{}_B \begin{bmatrix}AA_1\\BA_2\end{bmatrix} X^{A_1A_2A} + \ldots. \tag{4}$$

The tensor operators define finite-dimensional matrix representations of $\mathrm{gl}(m/n)$ when the right-hand sides of the transformation rules are rewritten as $X^{XYZ}(E^A{}_B)_{XYZ\ldots}{}^{CDE\ldots}$. For example, the matrices $(E^A{}_B)_X{}^Y = \delta^A{}_X \delta_B{}^Y$ certainly satisfy the required rules (1). The tensor operator formalism merely provides a convenient way of handling these representations, and taking account of the grading of the representation carrier space.

The appropriate tool for handling the 'graded tensors' is the 'graded permutation groups'. Formally, a graded permutation of a string of objects (bosonic and fermionic) may be defined as a permutation, together with a sign factor whenever an odd number of fermionic objects is interchanged. Specifically, if $X^{A_1\cdots A_r}$ is a tensor of rank r, a graded permutation $\tilde{\pi}$ acting on X yields another tensor, with sign factor and permuted components

$$(\tilde{\pi}X)^{A_1\ldots A_r}=\Bigg\{\prod_{\substack{i<j\\ \pi_i^{-1}\pi_j^{-1}}}[A_iA_j]\Bigg\}X^{A_{\pi 1}A_{\pi 2}\ldots A_{\pi r}} \tag{5}$$

where π is regarded as permuting the labels in positions 1 to r. The factor compares each pair of labels $i<j$ in the original ordering, and inserts a sign function (negative for two fermions and positive otherwise) whenever these labels will appear reversed $(\pi^{-1}{}_i>\pi^{-1}{}_j)$ in the final ordering.

Clearly, the graded permutation group is isomorphic to the ordinary one; indeed, as shown in the appendix,

$$[(\widetilde{\rho\sigma})X]^{A_1\ldots A_r}=[\tilde{\rho}(\tilde{\sigma}X)]^{A_1\ldots A_r}. \tag{6}$$

Its utility for the graded tensors lies in the fact that the graded permutations commute with the action of the algebra. That is, if we define

$$(\delta X)^{A_1\ldots A_r}=[E^A{}_B, X^{A_1\ldots A_r}] \tag{7}$$

to be the change in X under the action† of $E^A{}_B$, then as shown in detail in the appendix,

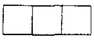
$$[\tilde{\pi}(\delta X)]^{A_1\ldots A_r}=[\delta(\tilde{\pi}X)]^{A_1\ldots A_r}. \tag{8}$$

These properties ensure that projection operators onto invariant tensors of definite graded symmetry type may be constructed as a product of column and row graded symmetrisations and antisymmetrisations. For example, for rank one, two and three, we have

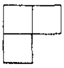
X_A

$S_{AB}=\frac{1}{2}(X_{AB}+[AB]X_{BA})$

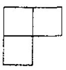
$A_{AB}=\frac{1}{2}(X_{AB}-[AB]X_{BA})$



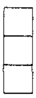
$$S_{ABC}=\tfrac{1}{6}(X_{ABC}+[AB][AC]X_{BCA}+[BC][AC]X_{CAB}+[AB]X_{BAC}+[BC]X_{ACB}+[AB][BC][AC]X_{CBA})$$



$$M^1_{ABC}=\tfrac{1}{3}(X_{ABC}+[AB]X_{BAC}-[AB][AC][BC]X_{CBA}-[AB][AC]X_{BCA})$$



$$M^2_{ABC}=\tfrac{1}{3}([BC]X_{ACB}-[AB]X_{BAC}+[AB][AC]X_{BCA}-[BC][AC]X_{CAB})$$



$$A_{ABC}=\tfrac{1}{6}(X_{ABC}+[AB][AC]X_{BCA}+[BC][AC]X_{CAB}-[AB]X_{BAC}-[BC]X_{ACB}-[AB][BC][AC]X_{CBA}).$$

These tensors possess graded versions of the usual symmetries and cyclic identities; for example,

$$A_{ABC}=-[BA]A_{BAC}=-[BC]A_{ACB}=[AC][BC]A_{CAB}$$

† Strictly the complete transformed operator $X'=\exp(+\theta E^A{}_B)X\exp(-\theta E^A{}_B)$ should be considered, where θ is an infinitesimal anticommuting parameter; however, δX in (7) is the essential quantity which must commute with $\tilde{\pi}\in\tilde{S}_r$.

$$M_{ABC} = -[AC]\begin{bmatrix} AB \\ C \end{bmatrix} M_{CBA}$$

$$M_{ABC} + [AB][AC]M_{BCA} + [BC][AC]M_{CAB} = 0.$$

The dimensions of such tensor representations may be obtained from the U(m)× U(n) reductions, which requires an analysis of the symmetries present with various combinations of even and odd indices. Clearly when indices are of one type, all even or all odd, the symmetry is simply that of the ungraded tableau or its transpose, respectively. The complete rule (see appendix) for $U(m/n) \supset U(m) \times U(n)$ is

$$\{\lambda\} = \sum_{\zeta} \{\lambda/\zeta\} \times \{\tilde{\zeta}\} \tag{9}$$

where the summation on ζ runs over all possible tableaux $\{\lambda/\zeta\}$ such that the product $\{\lambda/\zeta\}\,.\,\{\zeta\}$ (evaluated by the usual Littlewood–Richardson rule (Hamermesh 1962)) contains the tableau $\{\lambda\}$. For $U(m/1) \supset U(m) \times U(1)$, we have

$$\{\lambda\} = \sum_{k} \{\lambda/1^k\} \times \{k\}, \tag{10}$$

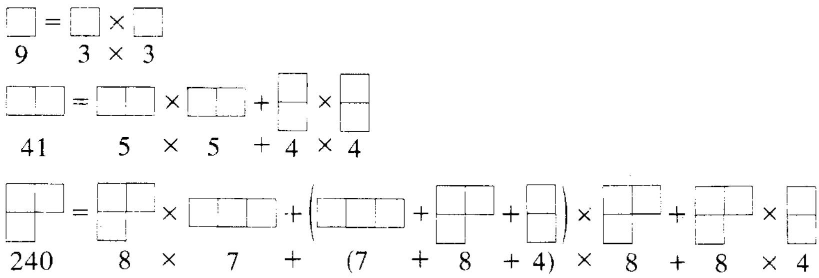
since the only non-vanishing U(1) tensors are the totally symmetrical ones with tableaux $\{k\}$. Using (9), the dimensions of the tensors of rank two and three may be written down in terms of m and n. We have

$$\begin{aligned}
&\square && = (m+n) \\
&\square\square && = \tfrac{1}{2}[m(m+1) + n(n-1) + 2mn] \\
&\begin{smallmatrix}\square\\ \square\end{smallmatrix} && = \tfrac{1}{2}[m(m-1) + n(n+1) + 2mn] \\
&\square\square\square && = \tfrac{1}{6}[m(m+1)(m+2) + n(n-1)(n-2) + 3mn(m+n)] \\
&\begin{smallmatrix}\square\square\\ \square\end{smallmatrix} && = \tfrac{1}{3}[m(m+1)(m-1) + n(n+1)(n-1) + 3mn(m+n)] \\
&\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix} && = \tfrac{1}{6}[m(m-1)(m-2) + n(n+1)(n+2) + 3mn(m+n)].
\end{aligned} \tag{11}$$

Note that the dimensions total $(m+n)^2$ and $(m+n)^3$, respectively, so that the tensors provide a complete decomposition of $\square \times \square$ and $\square \times \square \times \square$.

Diagrammatic rules similar to (9) and (10) govern branchings in other cases. For example, the reductions $U(m+\nu/n+\mu) \supset U(m/\mu) + U(\nu/n)$ and $U(m\mu + n\nu/m\nu + n\mu) \supset U(m/n) \times U(\mu/\nu)$ are given by (A.6) and (A.7). Furthermore, the usual product rule for Young diagrams operates also in this graded case (see appendix). Finally, just as in the $U(m) \supset SU(m)$ case (Hamermesh 1962), the tensors cannot be further reduced for $U(m/n) \supset SU(m/n)$, although some diagrams become identified, as will be seen below. The various branching rules remain the same for $SU(m/n) \supset SU(m) \times SU(n) \times U(1)$, with the U(1) weight readily identified from the reduction of the basic representation.

Some of the products and branching rules have been given for SU(2/1), SU(4/2) and SU(5/1) by Dondi and Jarvis (1979, 1980). Let us here illustrate (A.7) with the following reductions of SU(5/4) ⊃ SU(2/1) × SU(2/1):



and so on, the dimensions being given from (11).

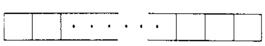
So far, we have considered only purely covariant tensors. From (3), however, there exist also purely contravariant tensors, which we can similarly reduce into graded symmetrised parts. Furthermore one can consider mixed co- and contravariant tensors, and it is easy to show that the trace of such a tensor with $\delta^A{}_B$ is invariant (the work of Jarvis and Green (1979) relied heavily on the representation of the Casimir invariants as traces of adjoint operators, namely $E^A{}_A$, $E^A{}_X[X]E^X{}_A, \ldots$). However, in the U($m$) and SU($m$) cases, the covariant and mixed tensors are related by modification rules to equivalent purely contravariant ones (King 1975). Thus the adjoint representation of SU(m) may be regarded as the Young diagram $\{2, 1^{m-1}\}$ or in the more convenient mixed form $\{\bar{1}, 1\}$, representing the traceless generators in two-index form, related by the rank-m alternating tensor, an SU(m) invariant. However, in the U(m/n) and SU(m/n) cases, there are no such invariant tensors or modification rules, and in general the contravariant, covariant and mixed tensors all correspond to inequivalent representations. By analogy with the U(m) and SU(m) cases, we shall associate contravariant tensors with Young diagrams with barred boxes.

Let us illustrate the foregoing with a simple case study, that of SU(2/1), comparing with other published results for this case (Scheunert *et al* 1977, Marcu 1980a, b), and foreshadowing some of the findings of the superfield analysis to follow in the next section. We define the weights in a representation as the eigenvalues of the diagonal Cartan subalgebra generators (cf Jarvis and Green 1979). Specifically, for U(2/1) these are $E^1{}_1, E^2{}_2$ and $E^3{}_3$, and for SU(2/1) we have, from (2), $A^1{}_1 = -E^2{}_2 - E^3{}_3$, $A^2{}_2 = -E^1{}_1 - E^3{}_3$ and $A^3{}_3 = (2E^3{}_3 + E^1{}_1 + E^2{}_2)$. The highest weight is defined in the sense of lexical ordering. For later convenience, we shall label irreducible representations not by the components of the highest weight, but by the eigenvalues denoted $(j, b)_{\mathrm{H}}$ of the generators

$$
\begin{aligned}
R_3 &= \tfrac{1}{2}(A^1{}_1 - A^2{}_2) = \tfrac{1}{2}(E^1{}_1 - E^2{}_2)\\
R_0 &= -\tfrac{1}{2}A^3{}_3 = -\tfrac{1}{2}(E^1{}_1 + E^2{}_2 + 2E^3{}_3)
\end{aligned}
\qquad (12)
$$

acting on the highest-weight vector.

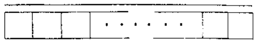
There are several subclasses of graded Young tableaux of SU(2/1), with differing types of SU(2) × U(1) content and dimension formulae. Consider, for example, the totally graded-symmetrical covariant tensor of rank $2j$ whose Young tableau consists of a single row of length $2j$:

$\{2j\} \sim$ 

The component $X_{11\ldots1}$ has highest weight $(2j, 0/0)$ in U(2/1). The irreducible representation of SU(2/1) is thus labelled $(j, -j)_{\mathrm{H}}$ and has SU(2) × U(1) decomposition

$$\{2j\}_{\mathrm{Y}}: (j, -j)_{\mathrm{H}} = (j)_{-j} + (j-\tfrac{1}{2})_{-j-\frac{1}{2}}, \tag{13}$$

and the dimension is $[2(2j)+1]$. The totally graded-symmetrical contravariant tensor of rank $2j+1$

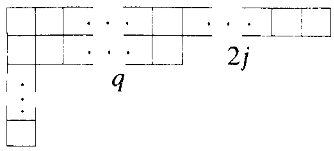
$\{\overline{2j+1}\} \sim$ 

has highest weight $(0, -2j/-1)$ in U(2/1) corresponding to the component $X^{322\ldots}$. The irreducible representation of SU(2/1) is therefore labelled $(j, j+1)$ with SU(2) × U(1) decomposition

$$\{\overline{2j+1}\}_{\mathrm{Y}}: (j, j+1)_{\mathrm{H}} = j_{j+1} + (j+\tfrac{1}{2})_{j+\frac{1}{2}} \tag{14}$$

and the dimension is $[2(2j+1)+1]$. These cases $\{2j\}$ and $\{\overline{2j+1}\}$ or $b_{\mathrm{H}} = -j_{\mathrm{H}}$ and $b_{\mathrm{H}} = j_{\mathrm{H}}+1$, correspond to the classes I, $\bar{\mathrm{I}}$ of Scheunert *et al* (1977).

Consider next the generic Young tableau

$\{2j+q+1, q+1, 1^p\} \sim$ 

The highest weight corresponds to the diagram with each box of the first two rows replaced by 1's and 2's respectively, and the remainder by 3's. The SU(2/1) label is $(j, b = -j-q-p-1)_{\mathrm{H}}$. From (9), the SU(2) × U(1) content is

$$\begin{aligned} &\{2j+q+1, q+1, 1^p\}_{\mathrm{Y}} \sim (j, -j-q-p-1)_{\mathrm{H}} \\ &(j, b)_{\mathrm{H}} = (j-\tfrac{1}{2})_{b-\frac{1}{2}} + j_b + j_{b-1} + (j+\tfrac{1}{2})_{b-\frac{1}{2}} \end{aligned} \tag{15}$$

and the dimension is $4(2j+1)$.

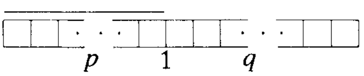
The results for the analogous contravariant (barred) case are

$$\{\overline{2j+q+1, q+1, 1^p}\}_{\mathrm{Y}} \sim (j, j+q+p+2)_{\mathrm{H}} \tag{16}$$

with identical dimension formula and SU(2) × U(1) content. For these cases, there are obviously modification rules like

$$\{2j+q+1, q+1, 1^p\} \sim \{2j+1, 1^{p+q}\}, \tag{17}$$

so that the label q is redundant (although the representations are in general inequivalent in U(2/1)). However, it is clear from (15) and (16) that the spectrum of b in these cases is $b \leqslant -j-1$ or $b \geqslant j+2$, respectively. In order to complete the spectrum for $-(j)+1 \leqslant b \leqslant (j+1)-1$, it is necessary to go to the traceless, mixed tensor with single contravariant and covariant rows:

$\{\bar{p}, q\} \sim$ .

The highest weight corresponds to the diagram with the q boxes replaced by 1's, and the p boxes by 2's, and the j and b labels are given by

$$\{\bar{p}, q\}_{\mathrm{Y}} \sim (\tfrac{1}{2}(p+q-1), \tfrac{1}{2}(p-q+1))_{\mathrm{H}}. \tag{18}$$

Moreover, since $p, q \geqslant 1$ and

$$\tfrac{1}{2}(p-q+1) = (\tfrac{1}{2}(p+q-1)+1) - q = -\tfrac{1}{2}(p+q-1)+p,$$

it is clear that these cases exhaust the range of b available to the irreducible tensors (the cases $b = j+1, b = -j$ being 'forbidden' since they have a different $SU(2) \times U(1)$ decomposition from (15)). The relations (15), (16) and (18) belong to class II of Scheunert *et al* (1977).

A plot of the irreducible tensor representations of SU(2/1) is given in figure 1, showing j_H (=spin content) against b_H. Obviously, the spectrum of b_H is discrete. This is clear in the case of the totally symmetrical tensors of class I, but in the case of class II, it is natural to expect the spectrum of b_H to be infinite and continuous. In the superfield realisations of the next section, we shall see that this is indeed the case, while the class-I irreducible representations correspond to special 'contrained' superfields.

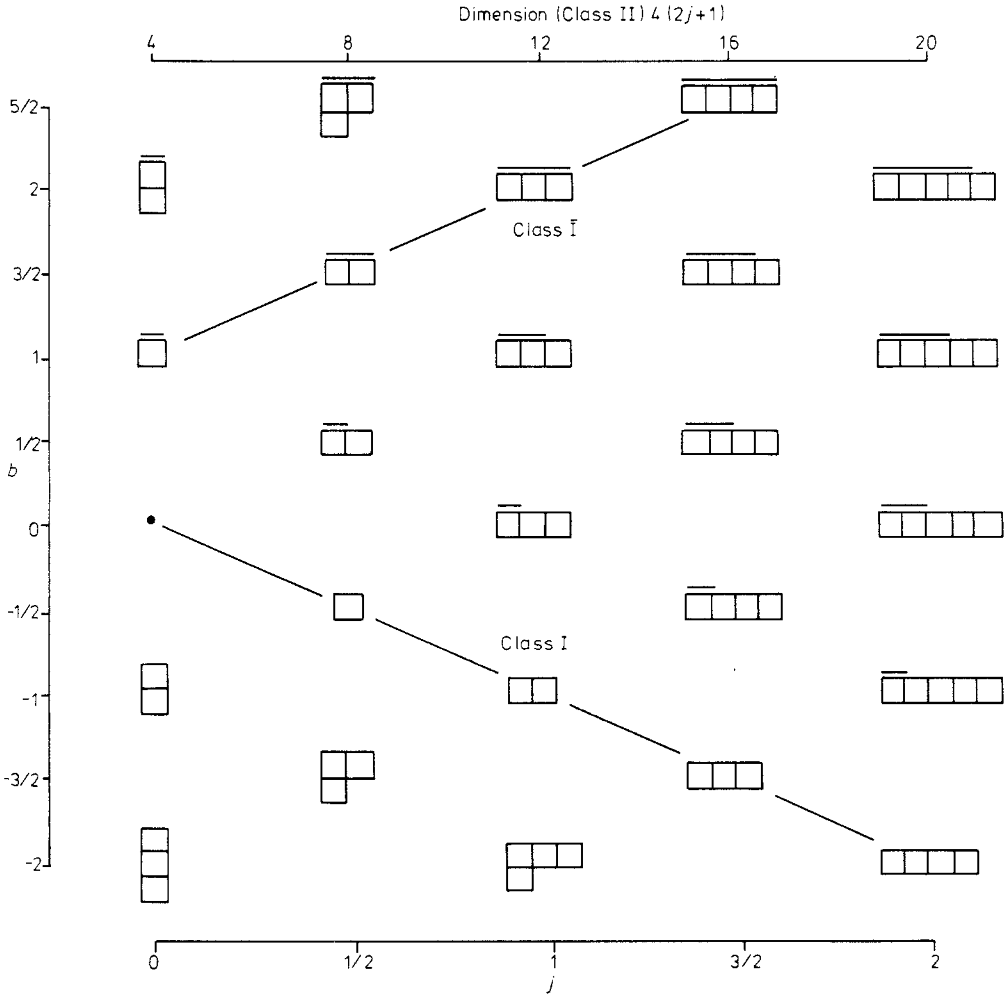


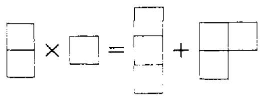
Figure 1. Spectrum of (j, b) for irreducible tensor representations of SU(2/1). Dimensions refer to class II only. The class-I tensors of rank l have dimension $(2l+1)$.

It is straightforward to see how the above results on the classification of SU(2/1) graded tensors generalise to SU(m/1). There are again classes I, $\bar{\text{I}}$ and II of irreducible representations, corresponding to barred or unbarred graded tableaux of $m-1$ rows or less, or general tableaux with at least m rows. The relations (13) and (14), for classes I and $\bar{\text{I}}$, generalise respectively to

$$\{\lambda\}_{\text{Y}} \sim \left(\lambda, -\frac{\Lambda}{2(m-1)}\right)_{\text{H}}$$
$$\{\bar{1}^p + \bar{\lambda}\}_{\text{Y}} \sim \left(\bar{\lambda}, \frac{\Lambda + mp}{2(m-1)}\right)_{\text{H}} \tag{19}$$

where $\Lambda = \Sigma\lambda_r$, and $\{\lambda\}$ has p rows. For class II, for fixed SU(m) content, including mixed tableaux with both barred and unbarred entries, b_{H} attains all half-integral values, except for the 'forbidden' values (19).

As was pointed out above, and demonstrated explicitly for □ × □ and □ × □ × □, the usual Littlewood–Richardson rule for evaluating Kronecker products in U(m), depending as it does only on the properties of the symmetric group, carries over to products of graded tensors in U(m/n) and SU(m/n). Examples in SU(2/1), with corresponding dimensions, are



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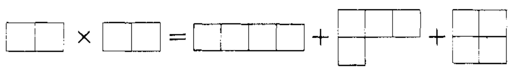
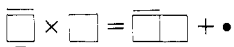


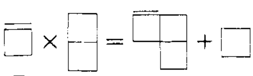
5 5 9 12 4



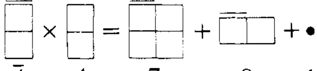
$\bar{3}$ 3 8 1

Obviously, Kronecker products of representations of the same type (barred or unbarred) remains of that type, and such products are completely reducible. This applies in particular to products of the form I × I and $\bar{\text{I}} \times \bar{\text{I}}$, but is more generally true. However, products such as I × $\bar{\text{I}}$ or II × II are likely to yield tableaux not occurring in the classification of irreducible tensor representations, if the row sum is too large. Apart from possible modification rules, such non-standard tableaux correspond to non-completely-reducible representations (which, however, have composition series with irreducible factors which may be isomorphic to standard tableaux).

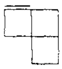
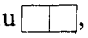
Examples of tensor products in SU(2/1) yielding non-completely-reducible representations are



$\bar{3}$ 4 9 3



$\bar{4}$ 4 7 8 1

It may be verified that  contains an invariant five-dimensional subspace with highest weight $(1, -1)_H$ associated with the tableau , while the four-dimensional factor space has highest weight $(0, 0)_H$, not associated with any of the standard tableaux of four dimensions with $j_H = 0$ (see figure 1). Similarly,  contains a six-dimensional invariant subspace with irreducible constituents of highest weight $(-\frac{1}{2}, \frac{1}{2})_H$ and $(0, 1)_H$, associated with  and , with a factor space corresponding to the trivial one-dimensional representation.

3. Superfields

For the construction of superfield representations, it is convenient to adopt a Cartesian basis for the superalgebras (1) and (2). We shall mainly consider the case of SU$(m/1)$, for which we define

$$Q_\alpha = A_\alpha^{\ m+1} \qquad \bar{Q}^\alpha = A^\alpha_{\ m+1}$$
$$R_0 = \tfrac{1}{2}\sum A^\alpha_{\ \alpha} = -\tfrac{1}{2}A^{m+1}_{\ m+1}$$
$$R_p = \tfrac{1}{2}(\lambda_p)_\alpha^{\ \beta}(A^\alpha_{\ \beta} - (2/m)\delta^\alpha_{\ \beta}R_0)$$

where λ_p, $p = 1, \ldots, m^2-1$ are the usual trace-normalised SU(m) matrices. Defining $(\lambda_0)_\alpha^{\ \beta} = \delta_\alpha^{\ \beta}$, and extending the SU$(m)$ structure constants to include $f_{pq0} = f_{p00} = f_{000} = 0$, the SU$(m/1)$ algebra is

$$\begin{aligned}
&[R_i, R_j] = \mathrm{i}f_{ijk}R_k\\
&[R_i, Q_\alpha] = -\tfrac{1}{2}(\lambda_i)_\alpha^{\ \beta}Q_\beta\\
&[R_i, \bar{Q}^\alpha] = +\tfrac{1}{2}\bar{Q}^\beta(\lambda_i)_\beta^{\ \alpha}\\
&\{Q_\alpha, \bar{Q}^\beta\} = (\lambda_p R_p)_\alpha^{\ \beta} - 2(m-1)/m(\lambda_0 R_0)_\alpha^{\ \beta}\\
&\{Q_\alpha, Q_\beta\} = \{\bar{Q}^\alpha, Q^\beta\} = 0
\end{aligned} \tag{20}$$

where $i, j, \ldots = 0, 1, \ldots, m^2-1$. The generalisation to SU(m/n) simply involves a second set of SU(n) matrices $\Lambda_1 \ldots \Lambda_{n^2-1}$, but will not be required in the following.

Superfields are constructed via the induced representation method, whereby in general one constructs a representation of a group G from a representation, say λ, of a subgroup H. Suppose the elements of H have matrix representations $[h]_a^{\ b}$ in some basis of the representation space V_λ. Consider the decomposition of G into its cosets G/H with some chosen coset representatives $x, y, \ldots$. The space of functions ϕ on G/H taking values in V_λ carries a representation of G if we define

$$g\phi_a(x) = [h^{-1}]_a^{\ b}\phi_b(z) \tag{21}$$

where zh is the unique decomposition of $g^{-1}x$ into a suitable coset representative and an element of H.

For supergroups, the natural choice of subgroup is the underlying Lie group itself. Thus for SU(m/n) we would have cosets labelled by anticommuting parameters $(\theta^{a\alpha}, \bar{\theta}_{a\alpha})$, corresponding to group elements $\exp[\mathrm{i}(\theta^{a\alpha}Q_{a\alpha} + \bar{Q}^{a\alpha}\bar{\theta}_{a\alpha})]$, and superfields

$\phi(\theta, \bar{\theta})$ over the full superspace. A simpler choice, for SU(m/n), are the (non-simple) subgroups $\bar{\Delta}(m/n)$ or $\Delta(m/n)$, generated by R_i and $\bar{Q}^{a\alpha}$ or $Q_{a\alpha}$, respectively, for which the coset representatives are just $\exp(i\theta^{a\alpha}Q_{a\alpha})$ or $\exp(i\bar{Q}^{a\alpha}\bar{\theta}_{a\alpha})$, corresponding to superfields $\phi(\theta)$ or $\phi(\bar{\theta})$.

Now an irreducible representation of SU$(m)\times$U(1)$\times$SU(n) has a natural extension to a representation of $\bar{\Delta}(m/n)$ or $\Delta(m/n)$ in which the generators of the Abelian subgroup, $\bar{Q}^{a\alpha}$ or $Q_{a\alpha}$, are mapped trivially to zero. We shall see the induced representations of this type (we shall speak of superfields of type (λ_E, b_E), where λ_E labels the SU(m) representation and b_E the U(1) weight, of the irreducible representation of SU$(m)\times$U(1)), provide realisations of the irreducible representations corresponding to the unbarred and barred graded Young tableau, and in general interpolate between the discrete spectrum of b values available in the tensor representations. In the SU(2/1) case, the superfields therefore yield all the irreducible representations. We conjecture that this is true for SU(m/n) also.

According to (21), the first stage in the construction of SU(m) superfields $\phi_a(\theta)$ and $\phi_a(\bar{\theta})$ of type (λ, b) is the evaluation of the left group action on cosets. From (20) we have, for infinitesimal group parameters r_p, r_0, η^α and $\bar{\eta}_\alpha$,

$$
\begin{aligned}
&\exp(iR_ir_i)\exp(i\theta^\alpha Q_\alpha)=\exp[i\theta^\alpha(\delta_\alpha{}^\beta-ir_i(\tfrac{1}{2}\lambda_i)_\alpha{}^\beta)Q_\beta]\exp(iR_ir_i)\\
&\exp(i\eta^\alpha Q_\alpha)\exp(i\theta^\alpha Q_\alpha)=\exp[i(\theta^\alpha+\eta^\alpha)Q_\alpha]\\
&\exp(i\bar{Q}^\alpha\bar{\eta}_\alpha)\exp(i\theta^\alpha Q_\alpha)=\exp[i\theta^\alpha(1+\theta\bar{\eta})Q_\alpha]\exp(\theta\lambda_p\bar{\eta}R_p-M\theta\bar{\eta}R_0)\exp(i\bar{Q}^\alpha\bar{\eta}_\alpha)
\end{aligned}
\tag{22}
$$

where $M=2(m-1)/m$. Thus, if $(t_p)_a{}^b$ and $b\delta_a{}^b$ are the matrices of the generators R_p and R_0 in the irreducible representation $(\lambda, b)_{\rm E}$ of SU$(m)\times$U(1), a superfield $\phi(\theta)$ of type (λ, b) has infinitesimal transformation laws

$$
\begin{aligned}
&\delta_{r_p}\phi_a(\theta)=ir_p(-t_p)_a{}^b\phi_b(\theta)-ir_p\theta^\alpha(\lambda_p/2)_\alpha^\beta\frac{\partial}{\partial\theta^\beta}\phi_a(\theta)\\
&\delta_{r_0}\phi_a(\theta)=-ir_0b\phi_a(\theta)-\tfrac{1}{2}ir_0\theta^\alpha\frac{\partial}{\partial\theta^\alpha}\phi_a(\theta)\\
&\delta_\eta\phi_a(\theta)=\eta^\alpha\frac{\partial}{\partial\theta^\alpha}\phi_a(\theta)\\
&\delta_{\bar{\eta}}\phi_a(\theta)=-(\theta\lambda_p\bar{\eta})(t_p)_a{}^b\phi_b(\theta)+M(\theta\bar{\eta})b\phi_a(\theta)+(\theta\bar{\eta})\theta^\alpha\frac{\partial}{\partial\theta^\alpha}\phi_a(\theta).
\end{aligned}
\tag{23}
$$

In (22) and (23), use must be made of the completeness relation of the matrices λ_p:

$$(\lambda_p)_\alpha{}^\beta(\lambda_p)_\gamma{}^\delta=2(\delta^\beta{}_\gamma\delta_\alpha{}^\delta-(1/m)\delta_\alpha{}^\beta\delta_\gamma{}^\delta).$$

As is well known in connection with space–time superfields, induced representations constructed in this way are of finite dimension because a component expansion in powers of θ must terminate. For SU$(m/1)$, the superfields $\phi_a(\theta)$ of type $(\lambda, b)_{\rm E}$ are of the form

$$\phi_a(\theta)=\phi_a+\theta^\alpha\phi_{a\alpha}+\ldots+\frac{1}{n!}\theta^{\alpha_1\alpha_2\cdots}\phi_{a[\alpha_1\ldots\alpha_n]}+\ldots. \tag{24}$$

From (23), the general transformation law of the nth component, with infinitesimal parameters r_i, η^α and $\bar{\eta}_\alpha$, is

$$\begin{aligned}\delta\phi_{a[\alpha_1\ldots\alpha_n]} &= \mathrm{i}r_p(-t_p)_a{}^b\phi_{b[\alpha_1\ldots\alpha_n]} + \mathrm{i}r_0(-b-\tfrac{1}{2}n)\phi_{a[\alpha_1\ldots\alpha_n]} \\ &\quad + \eta^{\alpha_{n+1}}\phi_{a[\alpha_1\ldots\alpha_n\alpha_{n+1}]} - \mathrm{i}r_p\{(\tfrac{1}{2}\lambda_p)_{\alpha_1}{}^\gamma\phi_{a[\gamma\alpha_2\ldots\alpha_n]} + \ldots\} \\ &\quad + (Mb+n-1)\{\bar{\eta}_{\alpha_n}\phi_{a[\alpha_1\ldots\alpha_{n-1}]} + \ldots\} - (t_p)_a{}^b\{(\lambda_p)_{\alpha_n}{}^\gamma\bar{\eta}_\gamma\phi_{a[\alpha_1\ldots\alpha_{n-1}]} + \ldots\}.\end{aligned} \tag{25}$$

Here the $\{\ldots\}$ stands for a sum of n terms antisymmetrising the indices $[\alpha_1\ldots\alpha_n]$.

The general superfield (24) is of dimension $2^m \times d_{\mathrm{E}}$, where d_{E} is the dimension of the $\mathrm{SU}(m)$ representation λ_{E} (with associated Young tableau $\{\lambda_E\}$). The $\mathrm{SU}(m)$ content is obviously given by the decomposition of the Kronecker products $\Sigma_{k=0}^m\{\lambda_{\mathrm{E}}\times 1^k\}$, or $\Sigma_{k=0}^m\{\bar{\lambda}_{\mathrm{E}}\times \bar{1}^{m-k}\}$, if we distinguish conjugate representations by barred tableaux. Now it is easily verified that for each k, $\{\lambda_{\mathrm{E}}\times 1^k\} = \{(1^m+\lambda_{\mathrm{E}})/1^{m-k}\}$, where $\{(1^m+\lambda_{\mathrm{E}})\}$ is the tableau obtained by adding a column of length m to $\{\lambda_{\mathrm{E}}\}$. Thus from (10), the $\mathrm{SU}(m)$ decomposition is identical to that of the $\mathrm{SU}(m/1)$ tensor representation labelled by the graded Young tableau $\{\lambda_{\mathrm{Y}}\} = \{(1^m+\lambda_{\mathrm{E}})\}$ or $\{(\bar{1}^m+\bar{\lambda}_{\mathrm{E}})\}$. The highest weight $(\lambda_{\mathrm{H}}, b_{\mathrm{H}})$ of the superfield is a component of $\phi_{a[\alpha_1\ldots\alpha_m]}$ and is therefore $(\lambda_{\mathrm{E}}, b_{\mathrm{E}}+\frac{1}{2}m)$ or $(\bar{\lambda}_{\mathrm{E}}, b_{\mathrm{E}}+\frac{1}{2}m)$. Although the superfield is equivalent to (i.e., has the same highest weight as) a symmetrised tensor representation of class II only if $b_{\mathrm{E}}+\frac{1}{2}m$ is half-integral, it is still useful to associate with it a generalised graded tableau having arbitrary continuous b.

We saw in (19) that for certain (discrete) values of b_{H}, the graded Young tableaux gave rise to representations of class I or $\bar{\mathrm{I}}$ with differing $\mathrm{SU}(m)$ structure. For these same b_{H} values, the corresponding superfield is decomposable: certain of the components form an invariant subspace. This is easiest to see for a scalar superfield, with λ_{E} the trivial representation. Consider, for example, the subspace of (24) with the components of order $(p+1)$ or higher zero, and the remainder non-zero. From (25), we have

$$\delta\phi^{p+1} \propto [Mb_{\mathrm{E}}+(p+1)-1]\phi^p.$$

The subspace will be invariant as claimed if Mb_{E} is equal to $-p$. The irreducible representation obtained is equivalent to the tensor of class I with graded tableau $\{1^p\}$, and highest weight $Mb_{\mathrm{H}} = -p/m$, as required by (19). These decompositions of the scalar superfield were demonstrated by Dondi and Jarvis (1980).

For a general superfield $(\lambda_{\mathrm{E}}, b_{\mathrm{E}})$, consider, for example, the subspace with ϕ^m and ϕ^{m-1} zero, except for the irreducible component $\{1^{m-1}+\lambda_{\mathrm{E}}\}$ of the latter. Writing

$$\phi_{a[\alpha_1\ldots\alpha_m]} = \epsilon_{\alpha_1\ldots\alpha_m}\phi_a$$

$$\phi_{a[\alpha_1\ldots\alpha_{m-1}]} = \epsilon_{\alpha_1\ldots\alpha_{m-1}\gamma}\phi_a{}^\gamma$$

in (25), we have (suppressing indices a and b)

$$\delta\phi \propto \bar{\eta}_\gamma[(Mb_{\mathrm{E}}+m-1)-(t_p\,.\,\lambda_p{}^{\mathrm{T}})]^\gamma{}_\beta\phi^\beta. \tag{26}$$

The only dangerous term involves the unconstrained component of $\phi_a{}^\beta$. However, on this subspace, the eigenvalue of $-(t_p\,.\,\lambda_p{}^{\mathrm{T}})$ is given in terms of the Casimir invariants as

$$-(t_p\,.\,\lambda_p{}^{\mathrm{T}}) = C_2\{1^{m-1}+\lambda_{\mathrm{E}}\} - C_2\{\bar{1}\} - C_2\{\lambda_{\mathrm{E}}\} \tag{27}$$

where

$$2C_2\{\lambda\} = \sum_{r=1}^m \lambda_r(\lambda_r+m+1-2r) - (\Lambda_{\mathrm{E}})^2/m$$

with

$$\Lambda_E = \sum_{r=1}^{m} \lambda_r.$$

From (26) and (27) we find

$$t_p \,.\, \lambda_p{}^T = -\Lambda_E/m$$

or (28)

$$Mb_E = -\Lambda_E/m - (m-1).$$

In terms of the highest weight $(\lambda_H, b_H) = (1^{m-1} + \lambda_E, b_E + \frac{1}{2}2(m-1))$, (28) becomes $Mb_H = -\Lambda_H/m$, are required by (19), for class-I representations.

A similar calculation ensues for the superfield $(\bar{\lambda}_E, b_E)$, and the subspace with ϕ^m, ϕ^{m-1} zero except for the irreducible component $\{\bar{\lambda}_E - \bar{1}^{m-1}\}$ of the latter. On this component,

$$t_p \,.\, \lambda_p{}^T = \Lambda_E/m + (m-1)$$

whence (29)

$$Mb_E = \Lambda^E/m.$$

The highest weight is $(\lambda_H, b_H) = ((\bar{\lambda}_E - \bar{1}^{m-1}), b_E + \frac{1}{2}(m-1))$, and (29) becomes $Mb_H = \Lambda_H/m + (m-1)$, in accord with (19), for class-$\bar{\text{I}}$ representations. The proofs, that (28) and (29) suffice to ensure the vanishing variation of the remaining components, can be completed similarly. The irreducible representations of SU$(m/1)$ so afforded correspond to the graded Young tableaux $\{1^{m-1} + \lambda_E\}$ and $\{\bar{\lambda}_E\}$, respectively.

There will obviously be several different choices of b_E capable of decomposing a given (non-scalar) superfield (λ_E, b_E). In all cases, such constrained superfields will correspond to one of the graded Young tableaux of class I or $\bar{\text{I}}$. We forego further details in favour of a complete investigation of the case $m = 2$, to which we now turn.

For SU(2/1), the superfield expansion (24) may be written (with external spin $j_E \equiv j$)

$$\phi_a(\theta) = A_a + \theta^\alpha B_{a\alpha} + \tfrac{1}{2}\theta^{\alpha_1}\theta^{\alpha_2}\epsilon_{\alpha_1\alpha_2}F_a.$$

Upon introducing the spin-$j \pm \frac{1}{2}$ projection operators $\pi^\pm$, defined by

$$\pi^+ = [(j+1) + \boldsymbol{t}\,.\,\boldsymbol{\sigma}]/(2j+1) \qquad \pi^- = (j - \boldsymbol{t}\,.\,\boldsymbol{\sigma})/(2j+1)$$

$$\boldsymbol{t}\,.\,\boldsymbol{\sigma} = j\pi^+ - (j+1)\pi^- m$$

the η, $\bar{\eta}$ component transformations (25) can be written

$$\delta A_a = \eta^\alpha B^+_{a\alpha} + \eta^\alpha B^-_{a\alpha}$$

$$\delta B^+_{a\alpha} = \pi^{+}_{a\alpha}{}^{b}\bar{\eta}_\beta F_b + (b-j)\pi^{+}_{a\alpha}{}^{b\beta}\bar{\eta}_\beta A_b$$

$$\delta B^-_{a\alpha} = \pi^{-}_{a\alpha}{}^{b\beta}\bar{\eta}_\beta F_b + (b+j+1)\pi^{-}_{a\alpha}{}^{b\beta}\bar{\eta}_\beta A_b$$

$$\delta F_a = -\bar{\eta}^\alpha(b+j+1)B^+_{a\alpha} - \bar{\eta}^\alpha(b-j)B^-_{a\alpha}.$$

Obviously a general (unconstrained) superfield has dimension $4(2j+1)$, corresponding to a class-II graded Young tableau $\{1^2 + 2j\}$ (but with arbitrary b). It can be seen, however, that two types of invariant subspace, I and $\bar{\text{I}}$, of $\{A_a, B^-_{a\alpha}, B^+_{a\alpha}, F_a\}$, arise: $\{A_a, 0, B^+_{a\alpha}, 0\}$ and label $(j, -j-1)_E$, or $\{A_a, B^-_{a\alpha}, 0, 0\}$ and $(j, j)_E$. The highest-weight labels are therefore $(j+\frac{1}{2}, -j-\frac{1}{2})_H$ and $(j-\frac{1}{2}, j+\frac{1}{2})_H$, corresponding respectively to the graded Young tableaux $\{2j+1\}$ and $\{\overline{2j}\}$. Correspondingly, the factor spaces by these invariant subspaces, with components of the form $\{\ldots, B^-_{a\alpha}, \ldots, F_a\}$ and $\{\ldots, \ldots, B^+_{a\alpha}, F_a\}$, have highest-weight labels $(j, -j)_{\mathrm{H}}$ and $(j, j+1)_{\mathrm{H}}$, again corresponding to class I and $\bar{\mathrm{I}}$, respectively (but with differing tableaux $\{2j\}$ and $\overline{\{2j+1\}}$).

4. Conclusions

The diagram and superfield techniques introduced generalise familiar techniques for constructing representations of the Lie superalgebras. It can be expected that similar methods will apply also to $\mathrm{osp}(m/n)$, $p(m)$ and $q(m)$, which we have not discussed in the present work. In these cases there is the possibility of projecting on to traceless subspaces by means of the appropriate metric, to provide a further decomposition of the symmetrised tensors.

The subject of Kronecker products could, of course, be pursued in the superfield framework. Here, products such as $\phi(\theta) \times \phi(\bar{\theta})$ give rise to noncompletely reducible representations. These properties obviously persist in the other superalgebras (except for the case of $\mathrm{osp}(1/n)$: see, for example, Corwin (1976)). However, the very simple nature of the composition series for the SU(2/1) examples mentioned suggest that here, too, diagrammatic methods may be appropriate.

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Appendix. Graded permutations

Here we prove the assertions of § 2 concerning the relationship between the graded permutation group, graded Young tableaux and tensor representations of the classical superalgebras.

Consider (6). The sign factor associated with the left-hand side is

$$\left(\prod_{\substack{i<j \\ \sigma^{-1}\rho^{-1}i > \sigma^{-1}\rho^{-1}j}} [A_i A_j] \right) \tag{A.1}$$

while that associated with the right-hand side is by definition

$$\left(\prod_{\substack{i<j \\ \rho^{-1}i > \rho^{-1}j}} [A_i A_j] \right) \left(\prod_{\substack{k<l \\ \sigma^{-1}k > \sigma^{-1}l}} [A_{\rho k} A_{\rho l}] \right).$$

Inserting separate products over $\sigma^{-1}\rho^{-1}i \lessgtr \sigma^{-1}\rho^{-1}j$, the first factor becomes

$$\left(\prod_{\substack{i<j \\ \rho^{-1}i > \rho^{-1}j \\ \sigma^{-1}\rho^{-1}i < \sigma^{-1}\rho^{-1}j}} [A_i A_j] \right) \left(\prod_{\substack{i<j \\ \rho^{-1}i > \rho^{-1}j \\ \sigma^{-1}\rho^{-1}i > \sigma^{-1}\rho^{-1}j}} [A_i A_j] \right) \tag{A.2}$$

and similarly changing k, l to $\rho^{-1}i, \rho^{-1}j$, and inserting separate products over $i \lessgtr j$ the second factor becomes

$$\left(\prod_{\substack{i<j \\ \rho^{-1}i<\rho^{-1}j \\ \sigma^{-1}\rho^{-1}i>\sigma^{-1}\rho^{-1}j}} [A_iA_j]\right)\left(\prod_{\substack{i>j \\ \rho^{-1}i<\rho^{-1}j \\ \sigma^{-1}\rho^{-1}i>\sigma^{-1}\rho^{-1}j}} [A_iA_j]\right). \tag{A.3}$$

The second term in (A.3) cancels the identical first term in (A.2), and the remaining terms may be combined, yielding (A.1) as required.

Equation (8) is proved most simply by taking the case of a transposition τ of adjacent labels, say A_i and A_{i+1}. The only terms of (8) which are not manifestly identical on the left- and right-hand sides are those involving substitutions of A_i and A_{i+1} by A in the action of $E^A{}_B$ on X. On the left-hand side these are, for example,

$$\ldots+[A_iA_{i+1}]\begin{bmatrix} AA_1 \\ B\,\vdots \\ A_{i-1}\end{bmatrix}\delta^{A_{i+1}}{}_BX^{A_1\ldots A_{i-1}AA_i\ldots A_r}+\ldots.$$

and on the right-hand side

$$\ldots+[AA_i]\begin{bmatrix} AA_1 \\ B\,\vdots \\ A_i\end{bmatrix}\delta^{A_{i+1}}{}_BX^{A_1\ldots A_{i-1}AA_i\ldots A_r}+\ldots,$$

with similar terms involving $\delta^{A_i}{}_B$. It can be verified in each case that, in the presence of the $\delta^{A_{i+1}}{}_B$ factor, the sign factors become identical, and (8) is proved for transpositions of adjacent elements. However, since any permutation may be expressed as a product of such transpositions, it is true in general.

Corresponding to an irreducible representation of the permutation group labelled by a Young tableau $\{\lambda\}$, the operator

$$\tilde{Y}^{\{\lambda\}}=(d/r!)\sum_{\pi}\bar{\chi}^{\{\lambda\}}(\pi)\tilde{\pi} \tag{A.4}$$

(where d is the dimension of $\{\lambda\}$ in S_r, and $\chi^{\{\lambda\}}$ the irreducible character), has the property of projecting onto invariant subspaces of the tensors of rank r of definite graded symmetry type $\{\lambda\}$, by analogy with the usual case (Hamermesh 1962). Obviously, if the tensor indices all happen to be even, the symmetry type is precisely $\{\lambda\}$. If the indices are all odd, the sign factor of $\tilde{\pi}$ is just the signature of π, the character of the antisymmetric representation $\{1^r\}$, and the symmetry type is that of the transposed tableau $\{\tilde{\lambda}\}$, since $\{\lambda\}\circ\{1^r\}=\{\tilde{\lambda}\}$ in S_r (Hamermesh 1962). Intermediate cases will have subsymmetries of the odd and even indices, depending upon $\{\lambda\}$ and the number of indices of each type.

The close analogy with the usual treatment of symmetrised tensors in the ungraded case extends to the reduction of products and the evaluation of branching rules, since these depend only upon the permutation group, and are unaffected by the presence of $\tilde{\pi}$, rather than π, in (A.4). Thus the Kronecker products in $\mathrm{U}(m/n)$ are governed by the usual (Littlewood–Richardson) rule for the outer Kronecker product (.) of representations of the symmetric group (Hamermesh 1962)

$$\{\lambda\}\,.\,\{\mu\}=\sum K^{\nu}_{\lambda\mu}\{\nu\} \tag{A.5}$$

where the $\{\nu\}$ are constructed by regular application of boxes. Conversely, the branching rule $\mathrm{U}(m+\nu/\mu+n) \supset \mathrm{U}(m/\mu)\times\mathrm{U}(\nu/n)$ is given by (cf King 1975)

$$\{\lambda\} = \sum_{\{\zeta\}} \{\lambda/\zeta\}\times\{\zeta\} \tag{A.6}$$

where the summation is over all possible $\{\zeta\}$, and $\{\lambda/\zeta\} = \Sigma\, K^{\lambda}_{\zeta\nu}\{\nu\}$, the usual division rule. For the special case $\mu = \nu = 0$, because the second factor relates to the symmetry of odd indices, the reduction (9) is found, which also yields the dimension formula for these symmetrised tensors.

For the branching rule $\mathrm{U}(m\mu+n\nu/m\nu+n\mu) \supset \mathrm{U}(m/n)\times\mathrm{U}(\mu/\nu)$, the relevant operation is that of the inner Kronecker product $(\circ)$ of representations of the symmetric group S_r. We have (Hamermesh 1962, King, 1975)

$$\{\lambda\} = \sum_{\sigma\in S_r} \{\lambda\circ\sigma\}\times\{\sigma\} \tag{A.7}$$

where r is the rank of $\{\lambda\}$.

It should be clear from our analysis that other diagram manipulations which depend only on the symmetric group (e.g. plethysms) also carry over to the graded case.

References

Bednar M and Sachl V 1978 *J. Math. Phys.* **19** 1487
Corwin L 1976 *Rutgers University Preprint*
Corwin L, Ne'eman Y and Sternberg S 1975 *Rev. Mod. Phys.* **47** 573
Dondi P H 1975 *J. Phys. A: Math. Gen.* **8** 1298
Dondi P H and Jarvis P D 1979 *Phys. Lett.* **84B** 75 (Erratum **87B** 403)
—— 1980 *Z. Phys.* C **4** 201
Dondi P H and Sohnius M 1974 *Nucl. Phys.* B **81** 317
Fairlie D B 1979 *Phys. Lett.* **82B** 97
Fayet P 1976 *Nucl. Phys.* B 113 135
Fayet P and Ferrara S 1977 *Phys. Rep.* **32** 69
Ferrara S, Wess J and Zumino B 1974 *Phys. Lett.* **51B** 239
Gates S J Jr, Stelle K S and West P C 1979 *Supergravity* ed P van Nieuwenhuizen and D Z Freedman (Amsterdam: North-Holland)
Grisaru M T 1977 *Phys. Lett.* **66B** 75
Hamermesh M 1962 *Group Theory and its Application to Physical Problems* (Reading, Mass: Addison-Wesley)
Ivanov E A and Sorin A S 1979 *JINR, Dubna, Preprints* E2-12363, 4
—— 1980 *J. Phys. A: Math. Gen.* **13** 1159
Jarvis P D 1976 *J. Math. Phys.* **17** 916
—— 1977 *J. Math. Phys.* **18** 551
Jarvis P D and Green H S 1979 *J. Math. Phys.* **20** 2115
Kac V G 1977 *Commun. Math. Phys.* **53** 31
Keck B W 1975 *J. Phys. A: Math. Gen.* **8** 1819
King R C 1975 *J. Phys. A: Math. Gen.* **8** 429
Marcu M 1980a *J. Math. Phys.* **21** 1277
—— 1980b *J. Math. Phys.* **21** 1284
Ne'eman Y 1979 *Phys. Lett.* **81B** 190
van Nieuwenhuizen P and Freedman D Z (eds) 1979 *Supergravity* (Amsterdam: North-Holland)
Rittenberg V 1978 *Lecture Notes in Physics* vol 79 ed. P Kramer and A Rieckers (Berlin: Springer)
Salam A and Strathdee J 1974 *Nucl. Phys.* B **76** 477
—— 1975 *Nucl. Phys.* B **84** 127
Scheunert M 1979 *Lecture Notes in Mathematics* (Berlin: Springer) p 716
Scheunert M, Nahm W and Rittenberg V 1977 *J. Math. Phys.* **18** 155

Sohnius M 1978 *Nucl. Phys.* B **138** 109
Taylor J G 1979a *Phys. Lett* **83B** 331
—— 1979b *Phys. Rev. Lett.* **43** 824
—— 1979c *Phys. Lett.* **88B** 291
Weyl 1939 *The Classical Groups* (Princeton, NJ: Princeton University Press)
Wouthuysen S A 1978 *Proc. Am. Inst. Phys.* **48**